Renormalizing Sznajd model on complex networks taking into account the effects of growth mechanisms

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Abstract. We present a renormalization approach to solve the Sznajd opinion formation model on complex networks. For the case of two opinions, we present an expression of the probability of reaching consensus for a given opinion as a function of the initial fraction of agents with that opinion. The calculations reproduce the sharp transition of the model on a fixed network, as well as the recently observed smooth function for the model when simulated on a growing complex networks.

1 Introduction

Opinions can either be made up by a person or taken over from another person. Sometimes some people try to force their opinions on others. In general, all people are free to form opinions as they see fit. The mechanism of opinion formation is "normative", i.e., normative in the sense of what *ought to be*, opposed to a "positive" mechanism, which is based on observation *what is* [1]. Based on this facts, and with the necessary simplifying assumptions, socio-physics gave the opportunity to apply techniques of statistical physics to model opinion formation among people [2–4].

One of the opinion formation models that has generated immediate interest in many authors on the field is the Sznajd model [5], which is based on the slogan "together we stand": individuals are represented by the lattice nodes (one-dimensional in its first version), and each randomly selected pair of neighbors convinces all their neighbors of their opinions, if and only if the pair shares the same opinion; otherwise, the neighbors' opinion are not affected. It differs from other consensus models by dealing only with communication between neighbors, and the information flows outward as in rumor spreading: a site does not follow what the neighbors tell the site.

On networks with fixed size, the results of the model do not depend much on the spatial dimensionality and type

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of neighborhood selected (i.e., two nodes convince the others, three convince the others, etc.) [6,7,9]. In the case of q choices of opinion, the model has q homogeneous absorbing states, where all individuals choose the same opinion; in the context of opinion, one says the system reaches consensus. The case of two opinions ($q = 2$) has been the most studied, denoting opinions as Ising variables "up" or $+1$, and "down" or -1. In more than one dimension, the probability (P_{up}) of reaching consensus "all up" depends on the initial fraction p of individuals with opinion "up"; for $p > 0.5$, the probability of reaching "all up" as stationary state is close to one, while for $p < 0.5$ it is negligible, having a sharp transition in $p = 0.5$, which can be interpreted as a dynamical phase transition. Computer simulations in [8] indicate that the universality class associated with this dynamical phase transition is different from the universality class of the Ising model. The distribution of time needed to reach the stationary state is a peak followed by a fast decay [10].

Much less is known about the Sznajd model on growing networks. Interactions of groups of people in some circumstances can be thought as a growing system, i.e., in a city with positive rate of immigration. In a first and simple approximation, it can be modeled by a growing scale-free network [11]. Recently, applying a Sznajd model recipe not after the complete network has been constructed, but while the network grows, i.e, while each new node is added to the network, one could observe that the Sznajd model simulated on scale-free networks, Barabási-Albert network
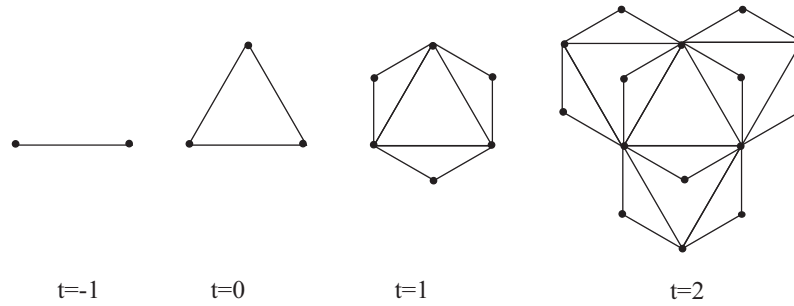
Fig. 1. The first four generations of the scale-free hierarchical graph. The growth start from a single edge connecting two vertices at $t = -1$. At each iteration step t, every edge generates an additional vertex, which is attached to the two vertices of this edge. The graph at $t + 1$ can be made by connecting together three t graphs.

and a hierarchical network [12], the system reaches consensus [13,14]. But in contrast to the sharp transition observed for the networks of fixed size, in which the Sznajd recipe is performed only after building up completely the network, the probability that the system reaches "all up" for a growing complex networks is a smooth function of p. In addition, this function depends on the type of neighborhood selected.

In this work, we propose a real space renormalization approach [16] to calculate the probability $P_{up}(p)$ of reaching consensus on opinion "up" as a function of the initial fraction p of opinion "up". Our results are for two common rules of neighborhood, namely "r-convince all their neighbors", with $r = 2$ and $r = 3$. We have obtained the two well-known results known for the model: a smooth function of p for the growing case and an expression which approximates the step function for fixed networks.

In the next section, we present the hierarchical network used in our calculations. Then, we present the renormalization approach and the analytical expressions obtained, each case is compared with the results from the numerical simulations, previously reported in [14], as well as for the BA scale-free network.

2 Hierarchical network

The deterministic scale-free graph used in this work grows as follows: It starts from a single edge connecting two vertices at $t = -1$. Thus, at $t = 0$, we have a triangle of edges connecting three vertices, at $t = 1$, the graph consists of 6 vertices connected by 9 edges, and so on (see Fig. 1). Note that the graph at $t + 1$ can be made by connecting together three t graphs. The total number of vertices at iteration t is

$$N_t = \frac{3(3^t + 1)}{2}. \tag{1}$$

This simple rule produces a complex growing network. The resulting graph is not a fractal, as is already pointed out in [12] the graph is surrounded by a long chain of edges and the resulting structure has not a fixed fractal dimension in contrast to, for example the perfect fractal generated in the Migdal-Kadanoff renormalization procedure [15]. In

the next section, we present the use of this hierarchical network to find expressions that agree with the simulated results of the Sznajd model on complex networks.

3 Renormalization approach

Our method can be very intuitive and is based on the method proposed by Galam to study bottom-up democratic voting by majority rule in a square lattice [3], where the predictions of the results in all the lattice are based on the applications of the majority rule over a basic cell of neighbors, called renormalization cell.

We find that given a neighborhood rule, it is enough to choose an appropriate generation of a hierarchical network for calculating $P_{up}(p)|_{r,g}$, which agrees with the the numerical results of the model on growing networks. The subscript index r, g in $P_{up}(p)|_{r,g}$ is to stress that the resulting function belongs to a chosen Sznajd rule (r) in a growing network (g). Subsequent self-iterations of $P_{up}(p)|_{r,g}$ result in a step function, i.e., $P_{up}^n(p)|_{r,g} = P_{up}(p)|_{r,f}$, where the subscript index f corresponds to the result obtained for a network of fixed size.

For a population fraction p with opinion "up", the general method is as follows:

– Given a neighborhood rule r, the chosen basic cell corresponds to the minimum generation t of the hierarchical network, such that $r > N_t$ (the r agents must have at least one agent to convince). We call this resulting number of nodes in the cell n_r.
– The probability of each possible configuration in a elementary cell is easily calculated, such that

$$1 = P_{all}(p)|_r = \sum_{k=2}^{n_r} B_{n_r k} p^k (1-p)^{n_r - k}. \tag{2}$$

with the binomial coefficient $B_{n_r k}$ over the appropriate cell for the chosen rule:

$$B_{n_r k} = n_r! / [k!(n_r - k)!]. \tag{3}$$

– From all the configurations calculated above, we select the subset that gives "all up" when applying the selected Sznajd rule on the cell, the sum of all of them is $P_{up}(p)|_{r,g}$:

$$P_{up}(p)|_{r,g} = P_{all}(p)|_{r,up} \qquad (4)$$

Next, we illustrate the result of the method with $r = 2$ and $r = 3$.

3.1 Case r = 2

3.1.1 Growing

For $r = 2$, the triangle of the generation $t = 0$ is the basic cell. Thus $n_r = 3$ and, for a given fraction p, all the possible configurations are:

$$1 = P_{all}(p)|_2 = p^3 + 2p^2(1-p) + 2p(1-p)^2 + (1-p)^3. \qquad (5)$$

If we apply the selected Sznajd rule $r = 2$ over the triangle, only the configurations expressed in the first two terms of the sum give "all up". Therefore:

$$P_{up}(p)|_{2,g} = p^3 + 2p^2(1-p) = 3p^2 - 2p^3. \qquad (6)$$

In Figure 2, we can see the good agreement of equation 6 with the numerical results [14] for the Sznajd model on a growing hierarchical, as well as for the Barabási-Albert scale-free network [11].

3.1.2 Fixed

In order to recover the reported result on a fixed network, one makes renormalization iterations, which means simply self-composing the equation (6):

$$P_{up}(p)|_{2,f} = P_{up}^{n_i}(p)|_{2,g}, \qquad (7)$$

and in the limit of large number of iterations ($n_i - 1$), one recovers the step function observed numerically for the model on fixed networks. Note that the number of terms and the coefficients sizes increase very fast, as one can observe in the expression of only one composition:

$$P_{up}^2(p)|_{2,g} = 27\,p^4 - 36\,p^5 - 42\,p^6 + 108\,p^7 - 72\,p^8 + 16\,p^9, \qquad (8)$$

therefore, the multiple compositions presented in Figure 2 are iterated with a computer. Figure 3 shows that the numerical simulations on large networks tend to the step function calculated from equation 8 with $n_i = 100$.

3.2 Case r = 3

3.2.1 Growing

The core of the method is the selection of the correct configurations after applying the Sznajd rule on it. As we will
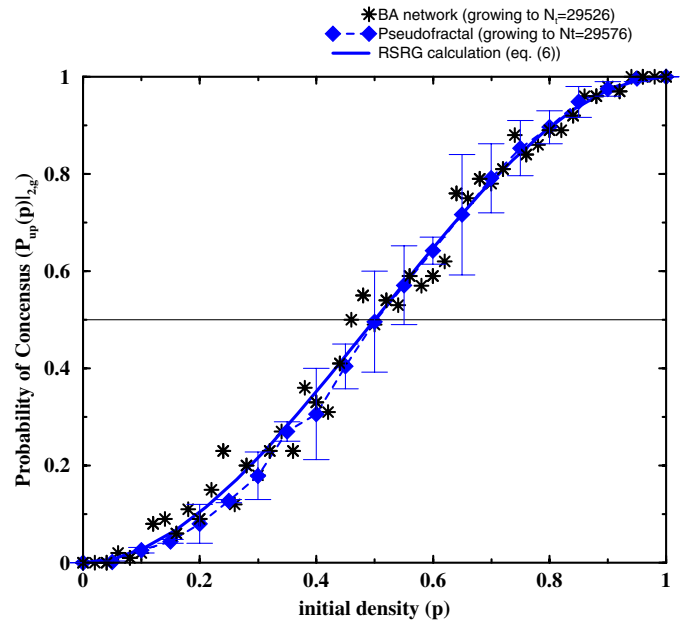

Fig. 2. Comparison between the function presented in equation 6 (solid line) with Monte Carlo simulations on a growing hierarchical network (triangles with error-bars) and on a growing BA scale-free network (stars). In both networks, 29 576 nodes are considered. We count the number of samples, out of 1000, for which the fixed point all "up" is obtained when different values for the initial concentration p of nodes "up" are simulated for rule $r = 2$.
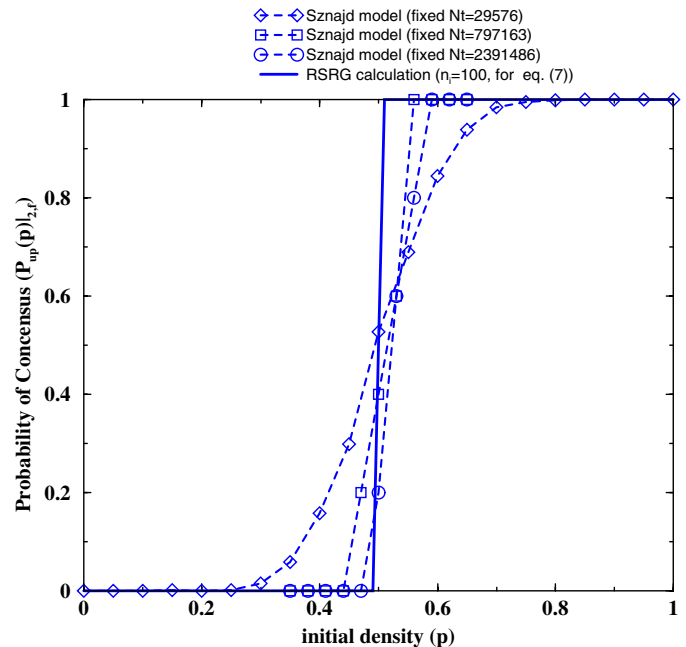

Fig. 3. Equation (7) with $n_i = 100$ (solid line) compared with simulations on a fixed hierarchical networks with $N_t = 29\,576$, 797 163 and 2 391 486 nodes (dashed line with symbols). Other simulation conditions as presented in the caption of Figure 2.
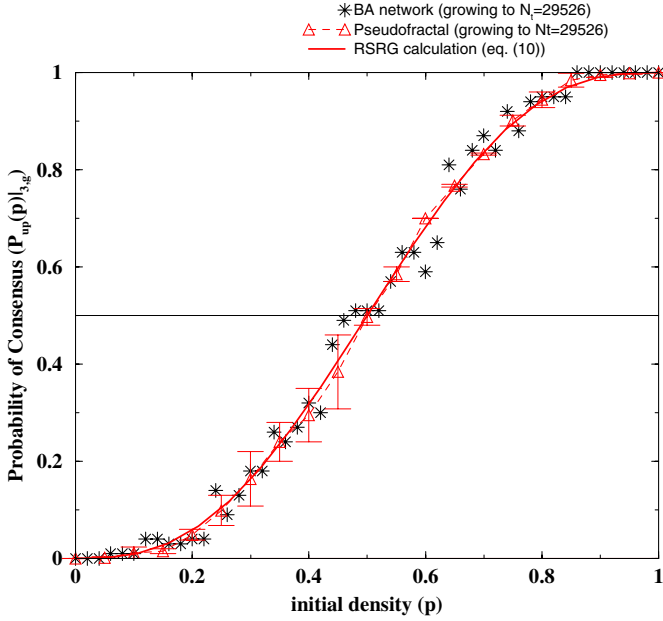
Fig. 4. Equation (10) (solid line) compared with the results from the simulations on a growing hierarchical network (triangles with error-bars) and on a growing BA scale-free network (stars) when $r = 3$. The other simulation conditions are the same of Figure 2.
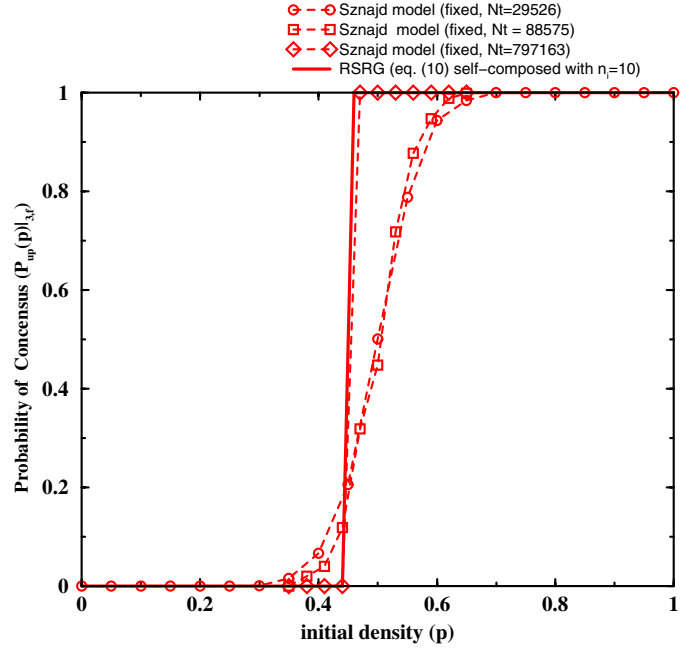

Fig. 5. Equation (10) self-composed in 9 steps (solid line) compared with the result from the simulations on a fixed hierarchical network networks with $r = 3$ (dashed lines with symbols). The other simulation conditions are the same as presented in the caption of Figure 3.

see for this rule, when the number of nodes in the renormalization cell is even, there are some symmetrical configurations which can have either "all up" or "all down" with the same probability. In this case only half of them are summed to P_{up}. For $r = 3$, the generation $t = 1$ is the basic cell. Thus $n_r = 6$ and, for a given fraction p, all the possible configurations are:

$$1 = P_{all}(p)|_3 = (1 + (1 - p))^6. \tag{9}$$

Note that the values of the binomial coefficient in the consecutive terms are: $1, 6, 15, 20, 15, 6, 1$. From the 20 configurations of the 4th term, there are 7 that give "all up" (shown in Fig. 6 at Appendix A), the corresponding 7 opposed cases which give "all down", and 6 symmetrical configurations shown in Figure 7 (Appendix A) that can give either "all up" or "all down". Therefore, these group of configurations contribute with $7 + 0.5 \times 6$, and we have:

$$P_{up}(p)|_{3,g} = p^6 + 6\,p^5\,(1-p) + 15\,p^4\,(1-p)^2 + 10\,p^3\,(1-p)^3. \tag{10}$$

In Figure 4, we see that equation (10) agrees very well with the numerical results [14] for the Sznajd model on a growing network when the rule $r = 3$ is considered.

3.2.2 Fixed

The result of the composition for this case is far more complicated and only 1 self-composition of equation (10)

($n_i = 2$) already needs a computer, as shows the following expression:

$$P_{up}(p)|_{3,g}^2 =$$
$$-1249989\,p^{12} + 390897\,p^{11} - 158184\,p^{10} + 28561\,p^9 -$$
$$643783179\,p^{18} + 270741222\,p^{17} - 100735317\,p^{16} + 41109081$$
$$p^{15} - 17504838\,p^{14} + 5585931\,p^{13} - 15244686567\,p^{24}$$
$$+ 11863411551\,p^{23} - 7642674243\,p^{22} + 4315583718\,p^{21}$$
$$- 2347570026\,p^{20} + 1281132990\,p^{19} - 816731505\,p^{30} +$$
$$2281401855\,p^{29} - 5100164190\,p^{28} + 9199907505\,p^{27} -$$
$$13440029166\,p^{26} + 15908268375\,p^{25} - 2187\,p^{36} + 65610\,p^{35}$$
$$- 925101\,p^{34} + 8148762\,p^{33} - 50268195\,p^{32} + 230706630p^{31}$$

In Figure 5 we see the step function obtained with only 9 steps of composition compared with the numerical results on a fixed network of different sizes; as we see the results agree very well with the simulations of the model on large networks.

4 Conclusions

Based on opinion formation rules of the usual Sznajd model, we use a renormalization approach to give an expression for the probability of consensus into one opinion as a function of the initial fraction of this opinion.
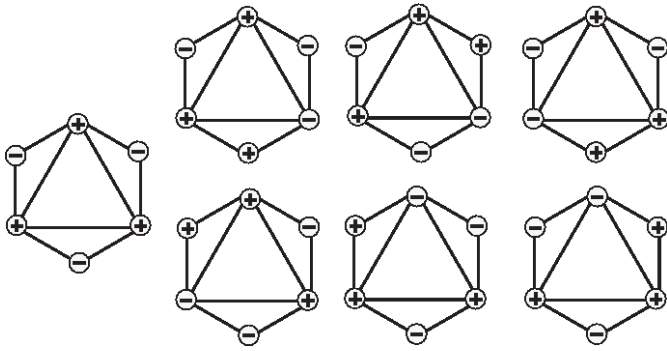
Fig. 6. Configurations that generate consensus "all up" with $r = 3$ and the same fraction of opinions "up" and "down".

Fig. 7. Configurations that generate either consensus "all up" or "all down" with $r = 3$.

We show that for a given Sznajd rule it is enough to solve exactly the model on an appropriate basic cell in order to find an expression for the smooth function, found numerically for the model on a growing network. Several self-compositions of the obtained function give the step function observed for the model on a network of fixed size. Further renormalization patterns has to be tested, but in order to reproduce the results of the Sznajd model on growing SF networks, a SF hierarchical network must be chosen.

The proposed method could be, in principle, extended to other types of neighborhood and more interestingly to many choices of opinion ($q > 2$), which is an feature of the model used to simulate elections processes [10,14,17], obtaining results consistent with some empirical observations [18].

Appendix A: Configurations with the same fraction of nodes "up" and "down"

Here we present some of the possible configurations applying the Sznajd rule corresponding to the $r = 3$ on its appropriate renormalization pattern ($n_3 = 6$). In particular, we show the case of half of the nodes having opinion up, mentioned in Section 3.2.1, and represented by the fourth term in equation (10).

Figure 6 shows the 7 configurations that give as a result "all up", when applying the Sznajd rule, i.e., three consecutive nodes with opinion $+1$ convince all their neighbors. Note that interchanging $+$ and $-$, we have the 7 configurations for the opposed case of consensus "all down".
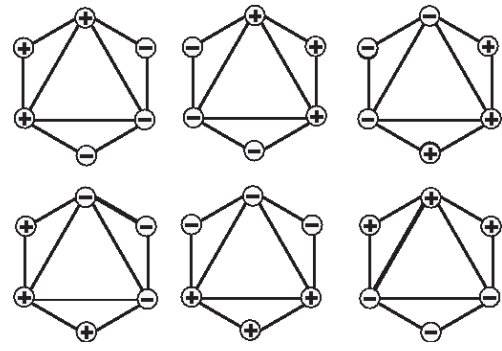
Figure 7 presents the 6 symmetrical configurations that have 3 consecutive nodes with $+1$, as well as 3 nodes with -1 giving consensus "all up" or "all down", respectively. Thus, these configurations contribute wit 0.5×6 to the probability of consensus "all up" (P_p), as showed in Section 3.2.1.

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